Symmetries of the Fokker-Planck equation and the Fisher-Frieden arrow of time

A. R. Plastino and A. Plastino*
Facultad de Ciencias Astronomicas y Geofisicas, Universidad Nacional de La Plata, Casilla de Correo 727, 1900 La Plata, Argentina
(Received 4 January 1996)

We give here a concrete illustration of Frieden's arrow of time, connected with Fisher's information measure. It is shown that this arrow of time is related to symmetries of the Fokker-Planck equation. [S1050-2947(96)05410-8]

PACS number(s): 05.40.+j

I. INTRODUCTION

In recent years, Frieden has shown that Fisher's information measure (FIM) can be regarded as an important tool for interesting developments in theoretical physics [1–8]. Among them, one may cite the possible existence of an ''arrow of time'' associated to this information measure, which following current usage we denote by I. Indeed, in discussing Brownian motion, Frieden [7] has put forward suggestive qualitative considerations regarding a putative FIM based ''*H-theorem.*''

Strong *indirect* evidence for the existence of an arrow of time has been put forward by Frieden [7]. (i) Assuming that Fisher's I decreases with time after a measurenment is made implies that I approaches a variational minimum [7]. Thus the information distance called $(I-J)$ in [7] approaches an extremum, which, in turn, leads to the relativistic quantum mechanics formalism (Eqs. (17)–(28) of Ref. [7]). (ii) Under the same assumption, several important physical laws can be rederived [1–7].

These logical developements constitute the motivation for the present search for a *direct* demonstration of the existence of a Fisher's information-based arrow of time. As far as we know, no analytical demonstration has been advanced showing that, for a system governed by a given probability distribution $W(\mathbf{x},t)$, (dI/dt) possesses a *definite* sign, so that an H theorem ensues and, with it, an ''arrow of time.'' In this report we show that the Fokker-Planck equation admits, under appropriate circumstances, of Fisher entropies (i.e., FIM's) that verify an H theorem. Moreover, it is seen that Fisher entropies whose temporal derivatives have a definite sign are related to symmetries of the Fokker-Planck (FP) equation.

II. THE FOKKER-PLANCK OPERATOR

We shall focus our attention on FP equations [9]

$$\frac{\partial W}{\partial t}=L_{FP}W, \tag{2.1}$$

where $W(\mathbf{x},t)$ is a normalized probability distribution

$$\int W(\mathbf{x},t)d^Nx=1, \tag{2.2}$$

$\mathbf{x}$ is a vector belonging to $\mathbb{R}^N$, and the FP operator [9] is given, in terms of a drift vector $\mathbf{V}_D$ of components $D_i(\mathbf{x},t)$ and of a diffusion tensor $\mathbf{D}$ of components $D_{ij}(\mathbf{x},t)$, by (Einstein convention used)

$$L_{FP}=-\frac{\partial}{\partial x_i}D_i(\mathbf{x},t)+\frac{\partial^2}{\partial x_i\partial x_j}D_{ij}(\mathbf{x},t). \tag{2.3}$$

The $D_{ij}(\mathbf{x},t)$ matrix is assumed to be a symmetric positive definite one, i.e.,

$$D_{ij}v_iv_j\geqslant 0, \tag{2.4}$$

for all vectors $\mathbf{v}$ in $\mathbb{R}^N$. It is convenient to introduce at this point the adjoint operator L^+_{FP} defined according to

$$L^+_{FP}=D_i\frac{\partial}{\partial x_i}+D_{ij}\frac{\partial^2}{\partial x_i\partial x_j}. \tag{2.5}$$

Given two probability distributions W_1 and W_2, our L operators verify

$$\int W_1(L_{FP}W_2)d^Nx=\int W_2(L^+_{FP}W_1)d^Nx. \tag{2.6}$$

III. FISHER ENTROPIES WHOSE TEMPORAL DERIVATIVES HAVE A DEFINITE SIGN

Associated with any pair of solutions $W_1(\mathbf{x},t)$ and $W_2(\mathbf{x},t)$ to the Fokker-Planck equation, we introduce the auxiliary quantity Q

$$Q=\int d^Nx W_1^2W_2^{-1}, \tag{3.1}$$

and immediately find

$$\frac{dQ}{dt}=2\int R(L_{FP}W_1)d^Nx-\int W_2(L^+_{FP}R^2)d^Nx, \tag{3.2}$$

where

$$R=\frac{W_1}{W_2}. \tag{3.3}$$

*Permanent address: Physics Department, National University La Plata, C.C. 727, 1900 La Plata, Argentina.
Electronic address: Plastino@venus.fisica.unlp.edu.ar

Equation (3.2) can be recast in the following fashion:

$$\frac{dQ}{dt} = -2\int R\frac{\partial}{\partial x_i}(D_i W_1) d^N x - \int W_2 D_i \frac{\partial (R^2)}{\partial x_i} d^N x + 2\int R \frac{\partial^2}{\partial x_i \partial x_j}(D_{ij} W_1) d^N x - \int W_2 D_{ij}\frac{\partial^2 (R^2)}{\partial x_i \partial x_j} d^N x, \tag{3.4}$$

so that, after integrating by parts the second term on the right-hand side (rhs) of (3.4), which allows for the cancellation of the first two terms on that side of the equation, we arrive at

$$\frac{dQ}{dt} = 2\int R\frac{\partial^2}{\partial x_i \partial x_j}(D_{ij} W_1) d^N x - \int W_2 D_{ij}\frac{\partial^2 (R^2)}{\partial x_i \partial x_j} d^N x = 2\int R\frac{\partial^2}{\partial x_i \partial x_j}(D_{ij} W_1) d^N x - 2\int W_1 D_{ij}\frac{\partial^2 R}{\partial x_i \partial x_j} d^N x - 2\int W_2 D_{ij}\frac{\partial R}{\partial x_i}\frac{\partial R}{\partial x_j} d^N x, \tag{3.5}$$

and thus, after cancelling the first two terms on the right-hand side [the second one must be first (twice) integrated by parts. Here we assume that in both cases the ''integrated'' parts vanish (see the Appendix)] to

$$\frac{dQ}{dt} = -2\int W_2 D_{ij}\frac{\partial R}{\partial x_i}\frac{\partial R}{\partial x_j} d^N x. \tag{3.6}$$

Now, since $W_2 > 0$ and, of course, D_{ij} is a definite positive matrix, we are in a position to assert that

$$\frac{dQ}{dt} \leqslant 0. \tag{3.7}$$

It is to be stressed that, in order to obtain the above inequality, the only assumption needed is (i) that both W_1 and W_2 verify the FP equation, and (ii) that $W_2 > 0$. Nothing is presupposed concerning either normalization or the sign of W_1.

Let us consider now a *family* of normalized probability distributions $W_\theta = W_\theta(\mathbf{x}, t; \theta)$, that depend upon a parameter θ, are of a definite positive character, and (all of them) verify the Fokker-Planck equation (2.1).

We differentiate the Fokker-Planck equation with respect to θ

$$\frac{\partial}{\partial \theta}\left(\frac{\partial W_\theta}{\partial t}\right) = \frac{\partial}{\partial \theta}(L_{FP} W_\theta), \tag{3.8}$$

and, as L_{FP} is a θ-independent linear operator, we have

$$\frac{\partial}{\partial t}\left(\frac{\partial W_\theta}{\partial \theta}\right) = L_{FP}\left(\frac{\partial}{\partial \theta} W_\theta\right), \tag{3.9}$$

i.e., $(\partial W_\theta / \partial \theta)$ is *itself* a solution to the Fokker-Planck equation (although it is not necessarily a normalized, positive definite one).

This is the point at which Fisher's information measure makes its appearance. We write it in the form

$$I_\theta = \int \frac{1}{W_\theta}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 d^N x, \tag{3.10}$$

and make reference to the definition (3.1) of Q together with the inequality (3.7). It is clear that, after identification of (i) W_1 with $(\partial W_\theta / \partial \theta)$ and (ii) W_2 with W_θ, one has

$$\frac{dI_\theta}{dt} = -2\int W_\theta D_{ij}\frac{\partial}{\partial x_i}\left(\frac{1}{W_\theta}\frac{\partial W_\theta}{\partial \theta}\right)\frac{\partial}{\partial x_j}\left(\frac{1}{W_\theta}\frac{\partial W_\theta}{\partial \theta}\right) d^N x, \tag{3.11}$$

so that [cf. Eq. (2.4)]

$$\frac{dI_\theta}{dt} \leqslant 0, \tag{3.12}$$

our main result.

IV. FISHER'S INFORMATION AND SYMMETRIES OF THE FOKKER-PLANCK EQUATION

Fisher's information I_θ is generally thought of in connection with the problem of estimating the actual value of the parameter θ [10], assuming the *form* of the appropriate probability distribution $W_\theta(\mathbf{x}, t, ; \theta)$ is known (although, of course, the precise value of θ is unknown) [10].

We will now *assume* that the Fokker-Planck equation (2.1) is symmetrical with respect to *some* change of variables. In this important case, given a solution $F_0(\mathbf{x}, t)$, it is possible to build up in a natural fashion, on account of that symmetry, a monoparametric family of solutions $F_\theta(\mathbf{x}, t, \theta)$. This means that for Fokker-Planck equations that admit this kind of symmetry, any solution belongs to a monoparametric family of solutions. The existence of such a symmetry is a *sufficient* condition in order to build up the desired family of solutions. Hence, *for any solution* the concomitant Fisher's information verifies the corresponding H theorem. Thus we assume (i) that a θ-dependent point transformation

$$x_i' = x_i'(x_1, \ldots, x_N; \theta) \quad (i = 1, \ldots, N), \tag{4.1}$$

exists such that, for $\theta = 0$ the identity

$$x_i = x_i'(x_1, \ldots, x_N; 0) \quad (i = 1, \ldots, N), \tag{4.2}$$

holds and (ii) that our Fokker-Planck equation is of such a type that, for any solution $F_0(\mathbf{x}, t)$, another solution $F_\theta(\mathbf{x}, t, \theta)$ exists given by

$$F_\theta(x_1, \ldots, x_N, t; \theta) = F_0(x_1', \ldots, x_N', t). \tag{4.3}$$

We expand now (4.1) in a θ power series

$$x_i' = x_i + \theta \eta_i(x_1, \ldots, x_N) + \cdots \quad (i = 1, \ldots, N), \tag{4.4}$$

where

$$\eta_i = \left(\frac{\partial x_i'}{\partial \theta}\right)_{\theta = 0} \quad (i = 1, \ldots, N), \tag{4.5}$$

and introduce the ''infinitesimal transformation generator'' [11]

$$\hat{X}=\eta_i\frac{\partial}{\partial x_i}. \tag{4.6}$$

It is seen that

$$\left[\frac{\partial F_\theta}{\partial\theta}\right]_{\theta=0}=\left[\frac{\partial x_i'}{\partial\theta}\right]_{\theta=0}\frac{\partial F_0}{\partial x_i}=\eta_i\frac{\partial F_0}{\partial x_i}=\hat{X}F_0, \tag{4.7}$$

so that the Fisher information measure associated to the parameter θ is given by

$$I_\theta[F_0]=\int\frac{1}{F_0}(\hat{X}F_0)^2d^Nx. \tag{4.8}$$

One gathers from (3.12) that, associated to any symmetry of the type prescribed by Eqs. (4.1), (4.2), and (4.3), a Fisher's entropy (4.8) exists that verifies an H theorem: its temporal derivative, for any Fokker-Planck solution F_0, is negative (or zero). Indeed, with reference to Eqs. (3.11) and (3.12), we are now in a position to assert that

$$\begin{aligned}\frac{d}{dt}I_\theta[F_0]&=\frac{d}{dt}\int\frac{1}{F_0}(\hat{X}F_0)^2d^Nx\\&=-2\int F_0D_{ij}\frac{\partial}{\partial x_i}\left(\frac{1}{F_0}(\hat{X}F_0)\right)\frac{\partial}{\partial x_j}\left(\frac{1}{F_0}(\hat{X}F_0)\right)d^Nx\\&\leqslant 0.\end{aligned} \tag{4.9}$$

We see that when the Fokker-Planck equation admits a symmetry related to a transformation of the form (4.1), we can compute dI/dt in terms of the infinitesimal transformation generator $\hat{X}$, which constitutes the second interesting result of the present communication.

V. EXAMPLES

A. Wiener process

We tackle here the one-dimensional (1D) instance with (i) null drift coefficient and (ii) constant diffusion coefficient $(D>0)$, i.e., the so-called Wiener process [9]. The associated Fokker-Planck equation adopts the form of a diffusion equation

$$\frac{\partial W}{\partial t}=D\frac{\partial^2 W}{\partial^2 x}. \tag{5.1}$$

Equation (5.1) admits of translational symmetry. Given any solution $W(x,t)$, a whole family of solutions $W(x+\theta,t)$ ensues, associated to the point symmetry

$$x_i'=x_i+\theta, \tag{5.2}$$

whose generator is given by

$$\hat{X}=\frac{\partial}{\partial x}. \tag{5.3}$$

The concomitant Fisher's entropy reads

$$I_\theta[W]=\int\frac{1}{W}\left(\frac{\partial W}{\partial x}\right)^2dx, \tag{5.4}$$

and, from Eq. (4.9),

$$\frac{dI_\theta}{dt}=-2\int DW\left(\frac{\partial^2(\ln W)}{\partial x^2}\right)^2dx\leqslant 0. \tag{5.5}$$

B. Liouville equation

The N-dimensional situation in which the diffusion tensor vanishes identically translates itself into the equation

$$\frac{\partial W}{\partial t}=-\frac{\partial}{\partial x_i}(D_iW), \tag{5.6}$$

that corresponds to the generalized Liouville equation describing the temporal evolution of an ''ensemble'' of solutions of the deterministic dynamical system [12]

$$\dot{x}_i=D_i(x_1,\ldots,x_N), \tag{5.7}$$

and has, of course, the form of an equation of continuity of flow.

It is clear that the present is a very important case, with manifold applications. Symmetries of the type prescribed by equations (4.1), (4.2), and (4.3) are associated to [cf. Eq. (3.11)] a *conserved* Fisher information measure

$$\frac{d}{dt}(I_\theta[W])=\frac{d}{dt}\int\frac{1}{W}(\hat{X}W)^2d^Nx=0. \tag{5.8}$$

The essential reason for the vanishing of the time derivative of Fisher's information can be here attributed to the fact that our tensor D fulfills the equality $D_{ij}=0$.

The behavior of I_θ is to be compared to that of Shannon's entropy

$$S=-\int W\ \ln W d^Nx. \tag{5.9}$$

It is known that [12]

$$\frac{dS}{dt}=\int d^Nx\frac{\partial D_i}{\partial x_i}. \tag{5.10}$$

Thus the sign of dS/dt depends upon the sign of the divergence of the vector $D_i(\mathbf{x})$. Hence, it is plain that dS/dt will grow with time only for those dynamical systems that have a positive phase flow divergence. If this is not the case, S may either remain constant, decrease [this happens in the case of *abstract dynamical systems* (non-Hamiltonian ones) characterized by a phase space flux with negative divergence. As a trivial example, consider the one-dimensional dynamical system $(dx/dt)=-x$, that possesses an atractor at the origin. It is clear that here the entropy of the pertinent ensemble of identical systems diminishes as time goes by] or exhibit a nonmonotonous temporal behavior [12]. All these types of behavior can take place in the case of *non-Hamiltonian dynamical systems.* In the particular case of Hamiltonian systems we have a divergenceless flow, so that $dS/dt=0$. Here we find the essentials of the well-known result that, for Hamiltonian systems in the fine-grained description, entropy does not change with time.

Moreover, S is only conserved in the case of divergenceless flows. On the other hand, the Fisher information measures associated to symmetries of the Liouville equation are *always* conserved (even for other types of flow). Thus Fisher's information may be useful in studying the behavior of solutions of the Liouville equation, because it provides one with conservation laws associated to symmetries of the equation.

A simple illustration is provided by the free particle in one dimension. We have

$$\dot{x}=\frac{p}{m}, \tag{5.11}$$

$$\dot{p}=0, \tag{5.12}$$

and the associated Liouville equation is

$$\frac{\partial W}{\partial t}=-\frac{p}{m}\frac{\partial W}{\partial x}. \tag{5.13}$$

Now, let $W(x,p,t)$ be a solution of this Liouville equation. It is apparent that $W(x+\theta,p,t)$ is also a solution. Thus we are in a position to write down [cf. Eq. (5.8)] the Fisher's entropy

$$I_1[W]=\int \frac{1}{W}\left(\frac{\partial W}{\partial x}\right)^2 dx \;\; dp=\text{const.} \tag{5.14}$$

VI. CONCLUSIONS

We have derived here two (related) results, namely, (i) to any family of probability distributions $W_\theta(\mathbf{x},t;\theta)$ that are solutions to the Fokker-Planck equation (2.1) we can associate a Fisher information measure I_θ whose temporal derivative has a definite sign [cf. Eq. (3.12)]. *This constitutes a concrete implementation of Frieden's arrow of time*.

More specifically, (ii) associated to any symmetry of the type prescribed by Eqs. (4.1), (4.2), and (4.3), a Fisher's entropy (4.8) exists that verifies an H theorem: its temporal derivative, for any Fokker-Planck solution F_0, is negative (or zero).

ACKNOWLEDGEMENT

The research of A. Plastino is partially supported by the National Research Council (CONICET) of Argentina.

APPENDIX

In the derivation of Eq. (3.6) we assume that for $|\mathbf{x}|\rightarrow\infty$, $W_{1,2}\rightarrow 0$ fast enough that

$$\int \nabla\cdot\left(\frac{W_1^2}{W_2}\mathbf{V}_D\right)d^N x=\lim_{r\rightarrow\infty}\int_S \frac{W_1^2}{W_2}\mathbf{V}_D\cdot\mathbf{da}=0, \tag{A1}$$

$$\int \nabla\cdot\left[W_1\mathbf{D}\cdot\nabla\left(\frac{W_1}{W_2}\right)\right]d^N x=\lim_{r\rightarrow\infty}\int_S W_1\mathbf{D}\cdot\nabla\left(\frac{W_1}{W_2}\right)\cdot\mathbf{da}=0, \tag{A2}$$

$$\int \nabla\cdot\left(\mathbf{U}\frac{W_1}{W_2}\right)d^N x=\lim_{r\rightarrow\infty}\int_S \mathbf{U}\frac{W_1}{W_2}\cdot\mathbf{da}=0, \tag{A3}$$

where $\mathbf{U}$ stands for the N-dimensional vector of components $\partial(W_1 D_{ji})/\partial X_j$, $i=1,\ldots,N$, S is an $(N-1)$-dimensional hypersphere of radius r centered at the origin and ($\mathbf{da}$ is the associated differential surface element).

The boundary conditions (A.1)–(A.3) are quite reasonable in the case of exponential solutions to the FP equation. In general, we can expect solutions of the form (for the sake of simplicity we discuss the 1D case)

$$W(x,t,\theta)=f(x,t,\theta)e^{-g(x,t,\theta)}, \tag{A4}$$

where θ is the parameter (alredy introduced) characterizing a monoparametric family of solutions. Remembering the derivation of Eqs. (3.11)–(3.12), we have

$$\frac{W_1}{W_2}=\frac{1}{f}(f'-fg'), \tag{A5}$$

$$\frac{W_1^2}{W_2}=\frac{1}{f}(f'-fg')^2, \tag{A6}$$

where the ''prime'' stands for partial differentiation with respect to the parameter θ. If we assume that the functions f and g are, for instance, polynomials, we see that the integrands in Eqs. (A.1)–(A.3) have the form of a rational function times the decreasing exponential e^{-g}, and the limits appearing in the rhs of those equations vanish. A similar argument can be given if we have solutions that are linear combinations of functions of the form (A.4). Note that the requirenments imposed by Eqs. (A.1)–(A.3) are quite similar to those corresponding to the convergence of the integral defining Fisher' information, since

$$I_\theta=\int \frac{1}{f}(f'-fg')^2 e^{-g}. \tag{A7}$$

This means that Eqs. (A.1)–(A.3) do not seem to impose stronger conditions than the very existence of Fisher information's integral I_θ.

[1] B. R. Frieden, Phys. Lett. A **169**, 123 (1992).
[2] B. R. Frieden, in *Advances in Imaging and Electron Physics*, edited by P. W. Hawkes (Academic, Orlando, 1994), Vol. 90, pp. 123–204.
[3] B. R. Frieden, Physica A **198**, 262 (1993).
[4] B. R. Frieden and R. J. Hughes, Phys. Rev. E **49**, 2644 (1994).
[5] B. Nikolov and B. R. Frieden, Phys. Rev. E **49**, 4815 (1994).
[6] B. R. Frieden, Phys. Rev. A **41**, 4265 (1990).
[7] B. R. Frieden and B. H. Soffer, Phys. Rev. E **52**, 2274 (1995).
[8] A. R. Plastino and A. Plastino, Phys. Rev. E **52**, 561 (1995).
[9] H. Risken, *The Fokker-Planck Equation* (Springer, New York, 1989).
[10] H. Cramer, *Mathematical Methods of Statistics* (Princeton University, Princeton, NJ, 1946).
[11] H. Stephani, *Differential Equations: Their Solution Using Symmetries* (Cambridge University Press, Cambridge, 1990).
[12] L. Andrei, Phys. Lett. **111A**, 45 (1985).